

Fermionic Loops in Numerical Stochastic Perturbation Theory

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We discuss the inclusion of fermionic loops contributions in Numerical Stochastic Perturbation Theory for Lattice Gauge Theories. We show how the algorithm implementation is in principle straightforward and report on the status of the project.

1. Introduction

In recent years the Numerical implementation of Stochastic Perturbation Theory (NSPT) was introduced, which was able to reach unprecedented high orders in perturbative expansions in Lattice Gauge Theories (LGT). Till now the main limitation of the method has been the quenched approximation, an inclusion of fermionic loops contributions missing. We can now fill this gap. Let us first of all remind the basics of the method. NSPT [1] comes (almost for free ...) as an application of the Stochastic Quantization scheme of Parisi and Wu [2]. We briefly sketch it in the context of LGT. The Cornell group showed [3] that a Langevin equation for LGT can be formulated as

$$U_\mu(x; \tau + \epsilon) = e^{-F[U(\tau), \eta]} U_\mu(x; \tau) \quad (1)$$

where

$$\begin{aligned} F[U(\tau), \eta] &= \sum_i T^i F_i \\ &= \sum_i T^i (\epsilon \nabla_{x, \mu}^i S[U] + \sqrt{\epsilon} \eta^i). \end{aligned} \quad (2)$$

In the previous formulae $S[U]$ is the action for the gauge fields, ϵ is a time step and η is a gaussian noise, while $\nabla_{x, \mu}^i$ is a Lie derivative on the group, whose definition can be easily understood in terms of

$$f(e^{\alpha T} U) = f(U) + \alpha^i \nabla^i f(U) + O(\alpha^2). \quad (3)$$

Eq. (1) should be understood as follows. One has at hand the theory described by the action $S[U]$

and the goal is to compute observables in terms of the functional integral, in which the measure is dictated by $\exp(-S[U])$ (we adhere to an euclidean formulation). In Eq. (1) an extra dependence on a new parameter τ is imposed on the fields. This parameter can be thought of as a stochastic time in which an evolution takes places according to Eq. (1), which is stochastic due to the presence of η . Now the key points are the gaussian nature of η and the fact that the so called drift term in the F appearing in Eq. (1) is given by the equations of motion. One can show that because of that the Langevin equation describes a stochastic process whose (asymptotic) equilibrium distribution is given just by the measure one is interested in ($\exp(-S[U])$). Therefore one can trade expectation values with respect to the latter for means over the stochastic time evolution dictated by Eq. (1). In the case of pure gauge Wilson action

$$S_G = -\frac{\beta}{2N} \sum_P \text{Tr} (U_P + U_P^\dagger) \quad (4)$$

Eq. (2) reads

$$\sum_i T^i \nabla_{x, \mu}^i S_G[U] = \frac{\beta}{4N} \sum_{U_P \ni U_\mu(x)} (U_P - U_P^\dagger)_{\text{tr}} \quad (5)$$

(the subscript tr asks for the traceless part). One can recognize in Eq. (5) a pretty local nature, as expected. It is well known [2] that one can get a Stochastic Perturbation Theory from the Langevin equation. NSPT gets it by directly ex-

panding the fields as

$$\begin{aligned}
U_\mu(x) &= \exp[A_\mu(x)] \\
&= \exp\left[\sum_{k>0} \beta^{-k/2} A_\mu^{(k)}(x)\right] \\
&= 1 + \sum_{k>0} \beta^{-k/2} U_\mu^{(k)}(x).
\end{aligned} \tag{6}$$

Such an expansion has to be thought of as a formal series motivated by the dependence of the solution to Eq. (1) on the coupling ($\beta^{-k/2} \sim g$). As a consequence every quantity depending on the fields can now be given a similar power expansion such as (for example)

$$F[U] = \sum_{k>0} \beta^{-k/2} F^{(k)}. \tag{7}$$

Eq. (1) gets translated into a hierarchy of equations exactly truncable at any given order and suitable for a numerical integration on a computer. Each order of Perturbation Theory is now obtained from means over stochastic time evolution of suitable (maybe complicate) composite operators (just like those appearing in Eq. (7)). Notice that having decompactified the formulation in Eq. (6), divergencies in the non-gauge-invariant sector show up, which are cured by Stochastic Gauge Fixing.

2. Introducing fermions

In principle introducing fermions is straightforward. One needs to face a new measure in the functional integral of the form

$$e^{-S_G} \det M = e^{-(S_G - \text{Tr} \ln M)}. \tag{8}$$

We write M for the fermion matrix to recall that other actions have the same structure, for example the Faddeev—Popov action [4]. One would in principle simply need to replace

$$\begin{aligned}
\nabla_{x,\mu}^i S_G &\mapsto \nabla_{x,\mu}^i S_G - \nabla_{x,\mu}^i \text{Tr} \ln M = \\
&\quad \nabla_{x,\mu}^i S_G - \text{Tr} ((\nabla_{x,\mu}^i M) M^{-1}).
\end{aligned} \tag{9}$$

As a matter of fact, one has to face an inverse, *i.e.* non-locality. The Cornell group [3] gave to the

problem a solution which was in a sense a prototype for fermionic simulations. This amounts to rewrite in Eq. (2)

$$\begin{aligned}
F_i &= \epsilon \mathcal{F}_i + \sqrt{\epsilon} \eta_i \\
\mathcal{F}_i &= \left[\nabla_{x,\mu}^i S_G - \text{Re} (\xi_k^\dagger (\nabla_{x,\mu}^i M)_{kl} (M^{-1})_{ln} \xi_n) \right]
\end{aligned} \tag{10}$$

in which a summation over repeated indices is to be understood. One should also keep in mind that k, l, n are multi-indices. A new gaussian field has been introduced, normalized as $\langle \xi_i \xi_j \rangle = \delta_{ij}$. The evolution of the process will now also average over ξ , resulting in

$$\begin{aligned}
\langle \mathcal{F}_i \rangle_\xi &= [\nabla_{x,\mu}^i S_G - \text{Tr} ((\nabla_{x,\mu}^i M) M^{-1})] \\
&= \nabla_{x,\mu}^i [S_G - \text{Tr} (\ln M)]
\end{aligned} \tag{11}$$

which is exactly what one is interested in. Life is now pretty easier, since one simply needs to invert the fermion matrix on a source solving the system

$$M_{kl} \psi_l = \xi_k \tag{12}$$

in terms of whose solution the evolution is local:

$$\mathcal{F}_i = \left[\nabla_{x,\mu}^i S_G - \text{Re} (\xi_l^\dagger (\nabla_{x,\mu}^i M)_{ln} \psi_n) \right]. \tag{13}$$

It takes a little time to realize that life is in a sense even easier in NSPT. To understand why, one should first of all remember that also the fermionic matrix gets expanded as a power series

$$M = M^{(0)} + \sum_{k>0} \beta^{-k/2} M^{(k)}. \tag{14}$$

The inverse of such a matrix is easy to compute

$$\begin{aligned}
M^{-1} &= \sum_{k=0} \beta^{-k/2} M^{-1(k)} \\
&= M^{(0)-1} + \sum_{k>0} \beta^{-k/2} M^{-1(k)}.
\end{aligned} \tag{15}$$

The notation enlightens the fact “the zeroth-order of the inverse is the inverse of the zeroth-order”. Other orders are not difficult to compute, resulting out of a simple recursion

$$\begin{aligned}
M^{-1(1)} &= -M^{(0)-1} M^{(1)} M^{(0)-1} \\
M^{-1(2)} &= -M^{(0)-1} M^{(2)} M^{(0)-1}
\end{aligned} \tag{16}$$

$$\begin{aligned}
M^{-1(3)} &= -M^{(0)-1} M^{(1)} M^{-1(1)} \\
&\quad -M^{(0)-1} M^{(3)} M^{(0)-1} \\
&\quad -M^{(0)-1} M^{(2)} M^{-1(1)} \\
&\quad -M^{(0)-1} M^{(1)} M^{-1(2)} \\
&\quad \dots
\end{aligned}$$

An NSPT algorithm for fermions is now easy to implement in force of the four considerations we now proceed to make.

2.1. Locality

First of all, from the new random fields ξ define the (power-expanded) fields

$$\xi^{(j)} \equiv M^{-1(j)} \xi \quad (17)$$

in terms of which the $(l+j)$ th contribution to Eq. (10) reads

$$\xi_k (\nabla_{x,\mu}^i M)_{kl}^{(l)} \xi_l^{(j)} + (l \leftrightarrow j, \text{ if } l \neq j) \quad (18)$$

The expression of $(\nabla_{x,\mu}^i M)_{kl}^{(l)}$ is easy to work out. In order not to obscure the main point with trivial algebra, we do not write it down, only stressing that it is as local as its non-perturbative counterpart, as obviously expected.

2.2. An easy recursive formula

Having made the point that a simple recursive formula holds for Eq. (16), it is straightforward to notice that a similar relation holds for the computation of the fields $\xi^{(j)}$

$$\begin{aligned}
\xi^{(0)} &= M^{(0)-1} \xi \\
\xi^{(1)} &= -M^{(0)-1} M^{(1)} \xi^{(0)}
\end{aligned} \quad (19)$$

$$\begin{aligned}
\xi^{(2)} &= -M^{(0)-1} \left[M^{(2)} \xi^{(0)} + M^{(1)} \xi^{(1)} \right] \\
\xi^{(3)} &= -M^{(0)-1} \left[M^{(3)} \xi^{(0)} + M^{(2)} \xi^{(1)} + M^{(1)} \xi^{(2)} \right]
\end{aligned}$$

...

The message from Eq. (19) is quite clear: at every order only one proper inverse is needed and everything that is left comes out of a simple recursive relation in terms of already computed (local) quantities. On top of that, the main point still needs to be made as far as the matrix $M^{(0)-1}$ is concerned.

2.3. The call for an FFT

Being the 0th order, $M^{(0)-1}$ of course does not depend on the fields: it is actually the standard Feynmann propagator, which is diagonal in Fourier space. Notice that the latter observation holds for any fermionic matrix and also for the case of the Faddeev-Popov matrix [4]. With this respect the method is quite general, opening the way to multiple applications, among which Neuberger fermions, whose perturbative expansions have till now only been pioneered. Given the above observations, the obvious way to implement the construction of the fermionic contribution to the drift F is to go back and forth from Fourier space. This of course calls for an efficient *FFT*. The implementation we are working on is the 4- d version of the algorithm described in [5]. In general, one can always think of a multi-dimensional *FFT* as the result of subsequent applications of 1- d *FFT*. This approach is the obvious choice on an architecture such that of the *APE* family computers, in which case the algorithm basically amounts to a 1- d (local) *FFT* interlaced with transpositions¹. Implementing an *FFT* in the NSPT context also opens the way to quite interesting byproducts. For example, one can think of obtaining NSPT results directly in momentum space. Also the possibility of exploiting the so-called Fourier acceleration can be taken into account.

2.4. Dealing with different time scales

By inspecting Eq. (10) one in general does not expect the characteristic times to be the same for the pure gauge and the fermionic contributions. Given the overhead imposed by the inclusion of fermions it is important not to waste computing time. This is quite easy to do because in order to correct for the finite time step ϵ in the integration of the Langevin equation we adhere to the simplest recipe, that is Euler scheme plus extrapolation $\epsilon \rightarrow 0$. Now a first order prescription for the integration of an equation of the type

$$f'(t) = g(t) + h(t) \quad (20)$$

¹Notice by the way that the reason why we have deferred till now the implementation of fermions in NSPT is just the fact that an *FFT* is easier on *APEmille* than on *APE100*

in which different characteristic times are present is very easy to work out, a trivial example being

$$\begin{aligned} f(t + \epsilon) &= f(t) + \epsilon g(t) \\ f(t + 2\epsilon) &= f(t + \epsilon) + \epsilon g(t + \epsilon) + 2\epsilon h(t + \epsilon). \end{aligned} \quad (21)$$

Having said that, the building blocks representation for our algorithm is basically made out of three modules:

- Evolution by the pure gauge contribution to the drift F with a certain time step ϵ .
- Generation of the $\xi^{(j)}$; this is the only non-local piece of the computation, the non-locality being anyway traded for multiple applications of an *FFT*, after which every operation is trivial.
- Evolution by the fermionic contribution to the drift F with a certain time step ϵ' . This does not present any structural difference with respect to the first module.

Of course the second and the third modules are tied together and their balance to the first is fixed by the considerations sketched a few lines above.

3. Conclusions, *i.e.* our shopping list

As already said, the implementation of the program is in progress, so that our conclusions are mainly a shopping list of applications we are working on.

- The highest priority is given to the unquenched extension of the α^3 computation of the Lattice Heavy Quark residual mass, which is a very important building block in the determination of the b -quark mass. The quenched result (86.2(0.6)(1.0)) has already made quite an impact [6]. Being unquenched non-perturbative simulations available, the unquenched extension of Perturbation Theory is compelling.
- The new NSPT record is by now the tenth order computation of the basic plaquette [7], which strongly confirmed the IR Renormalon dominance picture. Such a picture

has a clear forecast for the effect of the inclusion of fermions, which is fixed by the change in the β -function coefficients. It is worthwhile to verify this. Notice that as a byproduct this calls for a high loop determination of the critical hopping parameter, which could then be compared to the non-perturbative computation.

- Obviously, applications could be devised for coefficients needed in various improvement programs. We are at the moment considering the possibility of extending one loop further the computation of the coefficients needed in the twisted-mass formulation of Lattice QCD [8].

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REFERENCES

1. See for example R. Alfieri, F. Di Renzo, E. Onofri and L. Scorzato, *Nucl. Phys. B* **578**, 383 (2000) and references therein.
2. G. Parisi and Wu Yongshi, *Sci. Sinica.* **24**, 35 (1981).
3. G.G. Batrouni, G.R. Katz, A.S. Kronfeld, G.P. Lepage, B. Svetitsky and K.G. Wilson, *Phys. Rev. D* **32**, 2736 (1985).
4. G. Burgio, F. Di Renzo, G. Marchesini, E. Onofri, M. Pepe and L. Scorzato, *Nucl. Phys. B (Proc. Suppl.)* **73**, 853 (1999).
5. T. Lippert, K. Schilling, F. Toschi, S. Trentmann and R. Tripiccion, *Int. Journ. of Mod. Physics C* **8**, 1317 (1997).
6. See V. Lubicz, these proceedings and F. Di Renzo and L. Scorzato, to be issued soon.
7. F. Di Renzo and L. Scorzato, to be issued soon.
8. See M. Della Morte et al, these proceedings.